On the analogy between the transport of vorticity and heat in laminar boundary layers

By N. RILEY
Department of Mathematics, Durham Colleges, Durham

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The analogy between the transport of heat and vorticity when the Prandtl number is unity is used to provide a simple complementary solution of the boundary-layer energy equation for plane flow. The solution is extended to apply to axisymmetric boundary layers by suitably stretching the co-ordinate normal to the wall. Several applications of the solution are discussed.

1. Introduction

When the Prandtl number σ is unity a particular integral of the energy equation for steady compressible boundary-layer flow is provided by the well-known relation

$$T_H = \text{const.}, \tag{1}$$

where $T_H = T + (u^2/2C_p)$ is the total temperature; here T is the temperature, u the velocity in the direction of the mainstream and C_p the specific heat at constant pressure. The solution (1) describes the temperature distribution in the boundary layer on a thermally insulated surface. Further, if the pressure is everywhere constant, again with $\sigma = 1$, a more general solution of the energy equation is $T_H = Au + B$ where A and B are constants. This solution describes the temperature distribution in the steady boundary layer on a flat plate maintained at a constant temperature; the solution also has application to jet flows.

A further solution, in simple form, of the energy equation in thermal boundary layers when both the Mach number and relative temperature variations are small, with $\sigma = 1$, may be obtained by considering the analogy between the transport of heat and vorticity. Thus, for plane flow with $\sigma = 1$, the equation for the vorticity $\boldsymbol{\omega} = \zeta\mathbf{k}$ and the energy equation, with the effects of viscous dissipation neglected, become

$$\partial\zeta/\partial t + \mathbf{v}.\nabla\zeta = \nu\nabla^2\zeta, \quad \partial T/\partial t + \mathbf{v}.\nabla T = \nu\nabla^2 T, \tag{2}$$

where $\mathbf{v} = (u, v, 0)$ is the velocity vector and ν the kinematic viscosity; the physical properties of the fluid are assumed to be constant. In the boundary-layer approximation we can then write as a solution of the second of equations (2)

$$T = A\,\partial u/\partial y, \tag{3}$$

where y is the co-ordinate normal to the wall. The solution can be extended to axisymmetric boundary layers by a suitable stretching of the co-ordinate y, and to fully compressible boundary layers in which the pressure is constant and the viscosity μ is proportional to the temperature. Several applications of the solution are discussed in § 3.

2. Equations of motion

In the boundary-layer approximation the momentum, continuity and energy equations for fully compressible axisymmetric flow, in which there is no swirling component of velocity, over a body of revolution are

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{4}$$

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial x}(\rho r u)+\frac{\partial}{\partial y}(\rho v) = 0, \tag{5}$$

$$\rho\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)-\frac{1}{C_p}\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}\right) = \frac{1}{\sigma}\frac{\partial}{\partial y}\left(\mu\frac{\partial T}{\partial y}\right)+\frac{\mu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2. \tag{6}$$

Here x is measured along the wall from the forward stagnation point of the body and y is measured normal to the wall, and $r = r(x)$ denotes normal distance from the axis of symmetry. If $r = 1$ the above equations become those for compressible, two-dimensional boundary-layer flow. The specific heat C_p and Prandtl number σ are assumed to be constant. The pressure p is related to the density ρ and temperature T by the equation of state

$$p = \rho R T, \tag{7}$$

where R is the gas constant.

The solution of equation (6) may be written as the sum of the particular integral, which for steady flow with $\sigma = 1$ is given by equation (1), together with complementary solutions $\overline{T}$ satisfying the equation

$$\rho\left(\frac{\partial \overline{T}}{\partial t}+u\frac{\partial \overline{T}}{\partial x}+v\frac{\partial \overline{T}}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial \overline{T}}{\partial y}\right). \tag{8}$$

If viscous dissipation effects may be neglected then the energy equation takes the form (8). It is with solutions of equation (8) that we shall be concerned. If the pressure is everywhere constant then one solution of equation (8) is $\overline{T} = Au$. We shall show that:

(i) if $\rho = \rho_\infty$ and $\mu = \mu_\infty$ are constant a further simple solution of (8) is

$$\overline{T} = \frac{A}{r}\frac{\partial u}{\partial y}; \tag{9}$$

(ii) if ρ and μ vary with temperature with $\mu \propto T$ and $p =$ const. then the solution analogous to (9) is

$$\overline{T} = \frac{A}{\rho r}\frac{\partial u}{\partial y}. \tag{10}$$

It will be noted that for plane flow the solution (9) is simply the vorticity on the boundary-layer approximation as indicated in § 1. The solution with r not constant and the solution (10) are derived from the basic solution (3) by a suitable modification of the co-ordinate y.

In order to obtain the results (9) and (10) we differentiate the momentum equation (4) with respect to y to give

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left\{\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right\}. \tag{11}$$

Using the equation of continuity (5), the second term of the left-hand side of (11) may be written as

$$\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}+\rho r v\frac{1}{\rho r}\frac{\partial u}{\partial y}\right) = \frac{u}{r}\frac{\partial}{\partial y}\left\{\frac{1}{\rho}\frac{\partial}{\partial x}(\rho r u)\right\}+\rho r v\frac{\partial}{\partial y}\left(\frac{1}{\rho r}\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial y}\left\{\frac{u^2}{\rho r}\frac{\partial}{\partial x}(\rho r)\right\}-\frac{1}{\rho}\frac{\partial \rho}{\partial t}\frac{\partial u}{\partial y}, \tag{12}$$

and consequently equation (11) becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho r}\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{1}{\rho r}\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{1}{\rho r}\frac{\partial u}{\partial y}\right) = -\frac{1}{\rho r}\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\frac{\partial p}{\partial x}+\frac{1}{\rho r}\frac{\partial}{\partial y}\left\{\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right\}. \tag{13}$$

In either of the cases (i) and (ii) above the pressure term in equation (13) vanishes and, remembering that in case (ii) $\rho\mu = \text{const.}$, comparison of (13) with (8) establishes the results (9) and (10).

For steady flow the results (9) and (10) are obtained more directly using von Mises' transformation. If the equations (4) and (8) are written in von Mises' form where (x, ψ) are chosen as new independent variables, ψ being the stream function for steady flow defined by

$$\rho r u = \partial\psi/\partial y, \quad \rho r v = -\partial\psi/\partial x, \tag{14}$$

then differentiation of the momentum equation with respect to ψ yields the desired results immediately.

The results (9) and (10) have thus been established and in § 3 below we consider some applications of these solutions to steady flows.

3. Applications

(i) *Flat plate*

Consider first the flow over a semi-infinite flat plate. In this case the pressure is constant and we may allow variations of ρ and μ with temperature provided that $\mu \propto T$. Suppose that $x = 0$ corresponds to the leading edge of the plate and that for $x > x_0$ the plate is thermally insulated so that $\partial T/\partial y$ vanishes at the wall. By multiplying equation (4) by u, adding this to (6) and integrating the resulting equation with respect to y it can be shown that, for $x > x_0$,

$$\int_0^\infty \rho u(T_H - T_{H\infty})\, dy = \text{const.}, \tag{15}$$

where $T_H = T + (u^2/2C_p)$ is the total temperature with the subscript ∞ denoting its free stream value. In this case the solution of equations (4) and (5) may be written

$$u = U_\infty f'(\eta), \quad \text{where} \quad f(\eta) = \psi(\rho_\infty \mu_\infty U_\infty x)^{-\frac{1}{2}}, \tag{16}$$

and $f(\eta)$ satisfies

$$\left.\begin{aligned} &f''' + \tfrac{1}{2}ff'' = 0,\\ &f(0) = f'(0) = 0; f'(\infty) = 1.\end{aligned}\right\} \tag{17}$$

Thus, from (15) and (16) and provided that $T_H - T_{H_\infty}$ does not change sign across the boundary layer at $x = x_0$, we have $T_H - T_{H_\infty} \sim x^{-\frac{1}{2}}$ showing how T_H assumes its final value T_{H_∞} in the boundary layer. Since the wall is thermally insulated we have then, from equations (10) and (16),

$$T_H - T_{H_\infty} \sim Ax^{-\frac{1}{2}}f''(\eta). \tag{18}$$

If the distribution of T_H is given at any section, say $x = x_0$, then the constant A may be determined from equation (15). If $T_H - T_{H_\infty}$ changes sign across the boundary layer at $x = x_0$ in such a manner that $A \equiv 0$ then T_H approaches its final value T_{H_∞} more rapidly than $x^{-\frac{1}{2}}$. The functions $f'(\eta)$ and $f''(\eta)$ are shown in figure 1.

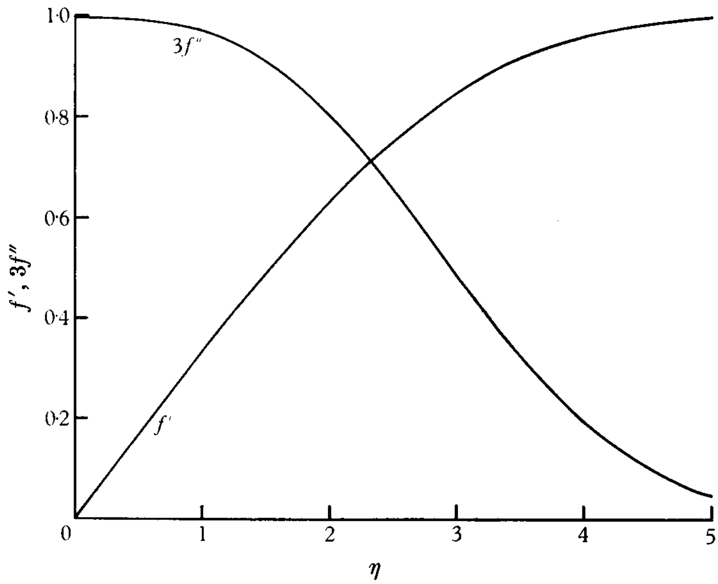

FIGURE 1. The functions f' and f'' for the boundary layer on a flat plate.

(ii) *Wedge flows*

Further simple examples of plane flow are provided by flows past symmetric wedges where the free-stream velocity $U_1 = ax^m$. In this case, with $m \neq 0$, the pressure is not constant everywhere and so, in accordance with § 2, we must have ρ and μ constant. An interesting special case is that of flow past a wedge of angle $\frac{1}{2}\pi$ giving $m = \frac{1}{3}$. To solve equations (4) and (5) we then put

$$\psi = (3a\nu)^{\frac{1}{2}} x^{\frac{2}{3}} f(\eta), \quad \eta = (a/3\nu)^{\frac{1}{2}} x^{-\frac{1}{3}} y, \tag{19}$$

and $f(\eta)$ satisfies

$$\left.\begin{aligned} &f''' + 2ff'' - f'^2 + 1 = 0, \\ &f(0) = f'(0) = 0; f'(\infty) = 1. \end{aligned}\right\} \tag{20}$$

We see from (19) that for this flow the solution (9) of equation (8) is the solution when the wall is maintained at a constant temperature T_w. Thus, if viscous dissipation effects can be ignored altogether, the temperature distribution in the boundary layer is given by

$$T = T_\infty + 0{\cdot}762(T_w - T_\infty)\, f''(\eta), \tag{21}$$

since $f''(0) = 1{\cdot}312$.

Another interesting application of the solution in this case is when the wedge is at yaw to the oncoming stream. The equation satisfied by the transverse component of velocity w is

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y} = \nu\frac{\partial^2 w}{\partial y^2}, \tag{22}$$

with $w = 0$ at $y = 0$, $w \to W_\infty$ as $y \to \infty$. The work of § 2 and equations (19) show that the required solution of equation (22) is

$$w = W_\infty\{1-0{\cdot}762f''(\eta)\}. \tag{23}$$

The functions $f'(\eta)$ and $f''(\eta)$ in this case are shown in figure 2.

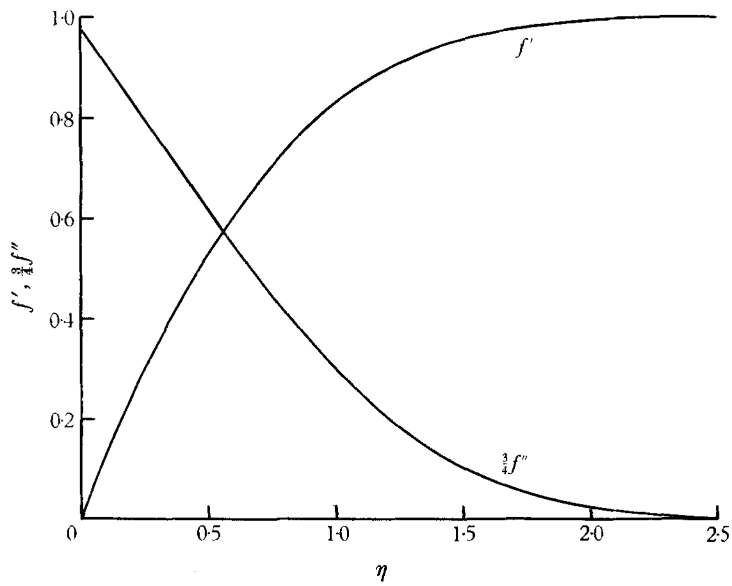

FIGURE 2. The functions f' and f'' for the boundary layer on a wedge of angle $\frac{1}{2}\pi$.

(iii) *Stagnation-point flow*

As an example of axisymmetric flow in which $r(x) = x$ we consider the case of stagnation-point flow. As is well known this flow is closely related to the previous example of plane flow and indeed the solution (9) is again the appropriate solution for a wall maintained at constant temperature. Because, if we write

$$\psi = (a\nu)^{\frac{1}{2}}x^2f(\eta), \quad \eta = (a/\nu)^{\frac{1}{2}}y, \tag{24}$$

then f satisfies equations (20) and again, from (9) and (24), we see that equation (21) is the solution for a wall maintained at constant temperature, if viscous dissipation effects are neglected.

(iv) *Jets*

With $dp/dx = 0$ and $r = 1$, equations (4) to (6) are appropriate to the flow in a two-dimensional free jet, or with $r(x) = x$ to its axisymmetric analogue the radial free jet. We shall confine our attention to the axisymmetric case, but the final results apply equally well to the two-dimensional flow. Since the pressure is constant we may allow variations of ρ and μ with T provided that $\mu \propto T$. To solve equations (4) and (5) we put

$$u = [(3M\rho_\infty\mu_\infty)^{\frac{2}{3}}/x\rho_\infty\mu_\infty]\, f'(\eta), \quad \text{where} \quad f(\eta) = \psi(3M\rho_\infty\mu_\infty x^3)^{-\frac{1}{3}}, \tag{25}$$

with $f(\eta)$ satisfying

$$\left.\begin{aligned} f''' + ff'' + f'^2 &= 0, \\ f(0) = f''(0) &= 0; f'(\infty) = 0, \end{aligned}\right\} \tag{26}$$

and

$$M = \int_0^\infty \rho x u^2\, dy$$

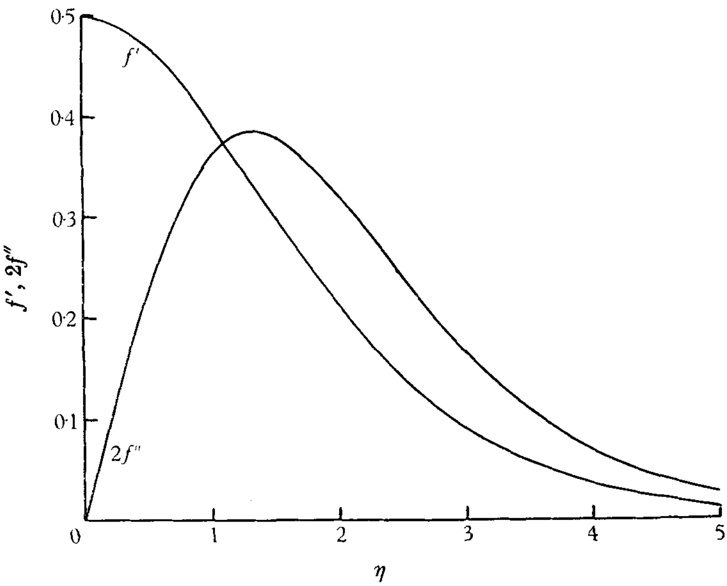

FIGURE 3. The functions f' and f'' for flow in a free jet.

is related to the radial flux of momentum in the jet. Symmetry about $\eta = 0$ has been assumed. Now the solution of (8), given by equation (10), namely

$$\overline{T} = (A/\rho x)\, \partial u/\partial y, \tag{27}$$

describes the effects on the temperature distribution of heating the jet in an anti-symmetrical manner whilst the other simple solution referred to,

$$\overline{T} = Au, \tag{28}$$

corresponds to the case when the heating is symmetrical. The functions $f'(\eta)$ and $f''(\eta)$ are shown in figure 3; the solution with $f(\infty) = 1$ has been chosen without loss of generality.